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The performance ratio of grouping policies for the joint replenishment problem*

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Abstract

In an EOQ model with n products, joint setup costs provide incentives for joint replenishments. These joint setup costs may be modelled as a positive, nondecreasing, submodular set function. A grouping heuristic partitions the n products into groups, and all products in the same group are always jointly replenished. Each group is thus considered as a single "aggregate product" being replenished independently of the other groups and according to the EOQ formula. As a result, possible savings when several groups are simultaneously replenished are simply ignored. The problem of determining the worst-case performance ratio of grouping policies is formulated as a maximin problem, which is neither quasiconcave nor quasiconvex. We use a novel approach to estimate an upper bound. We find that the cost of a best grouping policy is no more than 44.8% above the optimal cost.

1. Introduction

We consider a multiproduct extension of the traditional Economic Order Quantity (EOQ) model in which there is a cost incentive for simultaneous replenishment of several products. There are n products, and $N \triangleq \{1, \ldots, n\}$ denotes the set of products. Except for the setup costs (see below), the products are independent: there are no joint or dependent demands, no substitution opportunities, and products in N are not

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being used in making other products in N. As in the EOQ model, we make the following assumptions:

(1) We consider a continuous time, infinite horizon model, with stationary data (demand and cost rates) and no discounting. As a result, we focus on minimizing the long run average cost per time unit, while satisfying demands for all products.

(2) The *demand* for each product is deterministic and occurs at a constant rate. By rescaling the units for each product, we are assuming that the demand rate is of two units per time unit, for each product.

(3) The demand for each product is satisfied by continuously withdrawing from the inventory of that product. No shortages or backlogs are allowed. The inventories are replenished at times and in quantities to be determined. Replenishment of each product is instantaneous and lead times can be assumed to be zero, w.l.o.g. (without loss of generality).

(4) The *total cost* is the sum of all holding costs and setup costs.

(5) The holding cost for product i accumulates at a constant rate h_i dollars per unit of product and per unit time. For each product i ($i = 1, \dots, n$), rate h_i is a positive real number.

(6) At each replenishment, a positive *setup cost* $K(S)$ is incurred, depending only on the set $S \subseteq N$ of products being replenished.

In the traditional EOQ model, we have $K(S) = \sum_{i \in S} k_i$, where the k_i's are given separate setup costs. In this case, there is no incentive for joint replenishment, and an optimal policy has each product being independently replenished according to the familiar EOQ (or square root, or Harris's) formula.

We depart from the traditional EOQ model by allowing the joint setup cost $K(S)$ to be less than the sum of the separate setup costs of the products in S, therefore making joint replenishment costwise attractive. The model for joint setup costs we use here is that of a *submodular setup cost* introduced by Queyranne [12]. We assume the set function $K : 2^N \mapsto \mathbb{R}_+$ satisfies the following conditions:

(1) $K(\emptyset) = 0$;

(2) $K(S) > 0$, for all $S \subseteq N$, $S \neq \emptyset$;

(3) $K(S) \leq K(T)$, whenever $S \subseteq T$;

(4) $K(S \cup T) \leq K(S) + K(T) - K(S \cap T)$, for all S, T.

Conditions (1) and (2) are necessary for the model to be meaningful (otherwise optimal total costs for any finite period may go to $\pm\infty$). The nondecreasing property of condition (3) may be assumed w.l.o.g. (otherwise, set S is never used in an optimal solution, and function K can then be redefined so as to satisfy condition (3)). The *submodularity* property of condition (4) is fairly general, and allows the derivation of very tight bounds on the optimal cost. For example, the most popular joint setup cost function, defined by $K(S) \triangleq k_0 + \sum_{i \in S} k_i$ and sometimes called the major/minor, or quasilinear, or modular, setup cost function, is submodular. We refer to Goyal and Şatir [9] for a survey on models using this function. The "family model" of Roundy [16] and the example in Rosenblatt and Kaspi [14] are also submodular. We refer to Queyranne [12], Zheng [18] and Federgruen et al. [6] for further discussion of submodular setup costs. Section 2 below recalls how very tight bounds on the optimal cost may be obtained for submodular setup costs.

A *replenishment policy* is a specification of all replenishment epochs and quantities, for all products, over an infinite horizon. A *feasible policy* is a replenishment policy whereby all demands are satisfied, that is, inventory levels are never negative. The *joint replenishment problem* is to find a feasible policy with lowest possible long run average total cost per time unit. The only known fact about optimal policies is that it must satisfy the following *zero-inventory property*: the inventory of every product drops to zero just before any replenishment of that product. This property implies that replenishment quantities are directly implied by replenishment times (and conversely).

Much attention has focused on *stationary policies*, that is, policies where each product is replenished at constant time intervals (and therefore, by the zero-inventory property, in constant quantities). Nonstationary policies may have lower cost than any stationary policy (Andres and Emmons [1]). For submodular setup costs, however, it is known that the cost of a particular stationary policy (a "power-of-two" policy) cannot be more than 2% above the cost of any feasible policy, see Section 2.

One class of stationary policies that has been considered in the literature, is that of *grouping policies* (or fixed partition policies), whereby the set N of products is partitioned into *groups* of products, and all items in the same group are always jointly replenished. Each group is then considered as a single "aggregate product" being replenished independently of the other groups, and therefore according to the EOQ formula. As a result, possible savings when several groups are simultaneously replenished are simply ignored. Grouping policies are considered by Chakravarty et al. [4, 5], Rosenblatt and Kaspi [14] and Queyranne [13], for the joint replenishment problem, and by Page and Paul [11] and Anily [2] for related problems with budget or storage capacity linking the different products (see Gallego et al. [7], for an analysis of the latter problems).

An apparent advantage of grouping policies is that an optimal grouping may be relatively straightforward to compute. Chakravarty et al. show that, for special types of setup costs, the products can be indexed such that an optimal grouping is *consecutive*, that is, each group contains only products with consecutive indices. They also provide an $O(n^2)$ shortest path algorithm for finding a corresponding optimal grouping policy. Rosenblatt and Kaspi propose to find an optimal grouping by Dynamic Programming, for an arbitrary (not necessarily submodular) setup cost function. A correct Dynamic Programming algorithm runs in $O(3^n)$, Queyranne [13]. This might be acceptable when n is fairly small and the setup cost function is complicated, rendering other approaches (see Section 2) more cumbersome. Another apparent advantage of grouping policies is that they may be very easy to implement in practice, by permanently "tying together" all products in a same group.

In terms of cost, unfortunately, a best grouping policy can be somewhat worse than other feasible policies. An example in Zheng [18] shows that the cost of a best grouping policy can be worse than 20% above the cost of another feasible policy. The objective of this paper is to study, for submodular setup costs, how bad the best grouping policies can be in the worst case. We use submodular costs here because, as mentioned above, they provide a fairly general model and a very tight estimate on the optimal cost of any feasible policy is available for such setup cost functions.

The derivation of the upper bound in this paper turned out to be surprisingly difficult. Therefore, it may be of interest to outline here the method used in this derivation. First, we view the problem of determining the upper bound as a maximin optimization problem over the set of all possible instances: first to minimize over all grouping policies for any given instance, and then to maximize over all possible instances. Next, in Section 3, the original problem is simplified, through a reduction of the set of possible instances, to a nonlinear programming problem with $O(n)$ variables. However, this problem remains a difficult one to solve for the following reasons. First, its objective function is neither quasiconcave nor quasiconvex. It has many local minima and maxima, and the classic convex analysis methods are not sufficient for finding a global optimum. Second, this maximin problem cannot be solved by directly exchanging the min and max operators either. Third, the problem also has many constraints, which further complicate its resolution. Therefore, we estimate an upper bound on the optimal value instead of solving the problem exactly. To do so, we concentrate on a subproblem consisting of so-called "root-of-β path" instances. Although the worst-case performance ratio of grouping heuristics over this subproblem need not equal that over the original problem, we first show that the product of the performance ratio of grouping policies on this subproblem and the overall performance ratio of power-of-β policies (introduced in Section 2) yields an upper bound for the original problem. Then, a simple grouping heuristic is used for deriving an upper bound over the subproblem. This heuristic assigns at most two products to every group. Six inequalities are proposed, to simplify the estimation of an upper bound from this simple grouping heuristic. Finally, nonlinear programming techniques are applied to the approximation resulting from these six inequalities, and yield the requisite upper bound over the subproblem. This proof technique is admittedly rather elaborate, but it is the only one we know of which leads to a finite bound. With suitable modification, it may perhaps apply to a broader class of related inventory problems.

The organization of the paper is as follows. In Section 2, we recall the properties of a class of stationary policies, the *integer ratio policies*, distinct from the class of grouping policies. An analysis of the class of integer ratio policies allows the derivation of a very tight lower bound on the optimal cost of a feasible policy. This lower bound will be used in the definition of the performance ratio for grouping policies. A special class of integer ratio policies, the power-of-β policies, will be used later for deriving an upper bound on that performance ratio. Section 3 defines grouping policies and their performance ratio. Three lemmas in this section successively reduce the space of problem instances that has to be searched, and lead to a nonlinear programming formulation with only $O(n)$ variables. The last two lemmas in this section introduce a change of variables to be used in the next section. Section 4 gives the main result of this paper, an upper bound on the worst-case performance ratio of grouping policies, following the approach outlined above. Appendix A provides the proofs of the lemmas in the text. Appendix B collects, for reference, the notation used in this paper.

2. Lower bound on the average cost of all feasible policies

Besides grouping policies, another class of stationary policies has been widely studied, that is, *base period policies*, whereby all products are replenished at constant intervals ("cycles") which are integral multiples of a common *base period*. Base period policies are surveyed by Goyal and Şatir [9] for the case of major/minor setup costs.

Integer ratio policies are base period policies where, for any two products, the cycle of one product is an integer multiple of the cycle of the other one. A special class of integer ratio policies is that of *power-of-two* policies, where each interval is a power-of-two times the base period. An extension to *power-of-β policies*, where β is an arbitrary integer greater than one, is discussed below. Power-of-two policies were introduced for joint replenishment problems by Jackson et al. [10] and Roundy [15]. One of Roundy's major contributions was to show that finding best integer ratio policies also yields, as a by-product, a lower bound on the cost of *any* feasible policy.

Power-of-two policies, and the corresponding lower bound result, have also been extended by Roundy [16] to his "family model" of setup costs, and by Queyranne [12] and Zheng [18] to general submodular setup cost functions. For all these cases, the lower bound is also a very tight estimate (within about 2%) of the cost of a feasible policy.

As mentioned in the Introduction, this very tight bound is the main tool used here for assessing the performance ratio of grouping heuristics.

We now introduce the requisite notations and definitions. Let

$\mathbb{R}_+ \triangleq (0, \infty)$ be the set of positive real numbers;

$\mathbb{R}^n_+ \triangleq \times_{i=1}^{n} \mathbb{R}_+$ be the set of n-dimensional positive real numbers.

Given an integer $\beta \geq 2$, *a power-of-β policy* is defined by a vector $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n_+$ of product replenishment periods, satisfying the following three properties (Roundy [15]):

(1) For every product $i \in N = \{1, 2, \ldots, n\}$, an order is placed every $t_i > 0$ units of time, beginning at time zero.

(2) For some $\beta_0 \in [1/\sqrt{\beta}, \sqrt{\beta})$, we have $t_i = \beta^{m_i}\beta_0$ with $m_i \in \mathbb{Z}$ for all $i \in N$, where $\mathbb{Z} \triangleq \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers.

(3) The *zero-inventory property* holds, i.e., an order is placed for a product only when the inventory of that product drops to zero.

In the sequel, we denote a power-of-β policy by the associated replenishment vector t. For the joint replenishment model, we can easily derive a formulation of the average cost for a given power-of-β policy, see Queyranne [12], Zheng [18] and Federgruen et al. [6] for details. Indeed, a power-of-β policy t induces a permutation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the indices in N such that

$$t_{\alpha_1} \le t_{\alpha_2} \le \cdots \le t_{\alpha_n}. \tag{2.1a}$$

Let

$$U_i \triangleq \{\alpha_1, \alpha_2, \ldots, \alpha_i\}, \quad \forall i \in N \tag{2.1b}$$

be the sets associated with permutation α (ties are broken arbitrarily). Observe that under a power-of-β policy t, whenever product α_i is replenished, all the products in set U_{i-1} are also replenished. The average setup cost for power-of-β policy t is

$$K[t] = \sum_{i=1}^{n} K(U_i)\left(\frac{1}{t_{\alpha_i}} - \frac{1}{t_{\alpha_{i+1}}}\right) = \sum_{i=1}^{n} \frac{K(U_i) - K(U_{i-1})}{t_{\alpha_i}},$$

where $t_{\alpha_{n+1}} = \infty$, $U_0 = \emptyset$ and $K(\emptyset) = 0$.

The optimal average cost for power-of-β policies (for fixed β and β_0) is given by a nonlinear integer programming problem:

$$(\mathrm{JR})_\beta: \quad C_\beta(n, K, h, \beta_0) \triangleq \min_{t>0} \sum_{i=1}^{n} \left[\frac{K(U_i) - K(U_{i-1})}{t_{\alpha_i}} + h_{\alpha_i} t_{\alpha_i}\right],$$
$$\text{s.t.} \quad t_i = \beta^{m_i}\beta_0, \; m_i \in \mathbb{Z}, \; \forall i \in N,$$

where α satisfies (2.1a) and the U_i's are defined by (2.1b).

The following nonlinear programming problem (RJR), independent of β and β_0, is a continuous relaxation of $(\mathrm{JR})_\beta$.

$$(\mathrm{RJR}): \quad \mathrm{LB}(n, K, h) \triangleq \min_{t>0} \sum_{i=1}^{n} \left[\frac{K(U_i) - K(U_{i-1})}{t_{\alpha_i}} + h_{\alpha_i} t_{\alpha_i}\right],$$

where α satisfies (2.1a) and the U_i's are defined by (2.1b).

Sequence $(S_1, S_2, \ldots, S_q)$ of subsets of N is a *nested path* in N, if $\emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_q = N$. Note that the inclusions are proper in this definition. Note also that $(U_1, U_2, \ldots, U_n)$ defined above forms a nested path in N. Let

$$S^n \triangleq \left\{ K : 2^N \mapsto \mathbb{R}_+ \;\middle|\; \begin{array}{l} K(\emptyset) = 0; \\ K(S) \ge 0, \text{ and } K(N) > 0 \text{ (nonnegativity)}; \\ K(S) \le K(T), \text{ if } S \subseteq T \text{ (nondecreasing)}; \\ K(S \cap T) + K(S \cup T) \le K(S) + K(T), \\ \forall S, T \subseteq N \text{ (submodularity)} \end{array} \right\}$$

denote the set of submodular set functions on N.

The following characterization theorem from Zheng [18, Theorem 4.5] solves (RJR). It will be used in proving Lemma 3.1 (First reduction).

Theorem 2.1 (Optimal solution to (RJR)). *Assume that* $K \in S^n$ *and the components of* $t = (t_1, t_2, \ldots, t_n)$ *take on* q *distinct values* $t(1) < t(2) < \cdots < t(q)$, $q \le n$, *and* $(S_1, S_2, \ldots, S_q)$ *is a nested path in* N *with* $S_l \backslash S_{l-1} \triangleq \{i \in N \mid t_i = t(l)\}$. *Then* t *is optimal for* (RJR) *iff the following two conditions hold for each* $l = 1, 2, \ldots, q$:

(1) $t(l) = \sqrt{[K(S_l) - K(S_{l-1})]/h(S_l \backslash S_{l-1})}$, *where* $h(S) \triangleq \sum_{i \in S} h_i$, *for any set* $S \in N$.

(2) $\sqrt{[K(S) - K(S_{l-1})]/h(S \backslash S_{l-1})} \ge t(l)$, $\forall S$, *such that* $S_{l-1} \subseteq S \subseteq S_l$.

Besides, the optimal value of (RJR) *is*

$$\mathrm{LB}(n, K, h) = 2 \sum_{l=1}^{q} \sqrt{[K(S_l) - K(S_{l-1})]h(S_l \backslash S_{l-1})}.$$

Proof. See Zheng [18, Theorem 4.5]. $\square$

We introduce two sets of problem instances related to (K, h). They will be used in the sequel:

$$\mathcal{M}^n \triangleq \left\{ K : 2^N \mapsto \mathbb{R}_+ \,\middle|\, \begin{array}{l} 0 < K_1 \le K_2 \le \cdots \le K_n \text{ (nondecreasing);} \\ K(S) = \max\limits_{i \in S} K_i, \ \forall S \subseteq N \text{ (maximum)} \end{array} \right\}$$

denotes the set of "maximum" submodular set functions on N. Note that $K_i \triangleq K(\{i\})$, $\forall i \in N$.

$$\Omega^n \triangleq \left\{ (K,\ h) \in (\mathcal{M}^n, \mathbb{R}^n_+) \,\middle|\, \sqrt{\frac{K_1}{h_1}} < \sqrt{\frac{K_2 - K_1}{h_2}} < \cdots < \sqrt{\frac{K_n - K_{n-1}}{h_n}} \right\}$$

denotes the set of "monotone path" instances. Note the strict inequalities in the definition.

The following corollary follows immediately from Theorem 2.1.

Corollary 2.2 (Optimal solution to (RJR) for "monotone path" instances). *If* $(K, h) \in \Omega^n$, *then* $\mathrm{LB}(n, K, h) = 2 \sum_{i=1}^{n} \sqrt{(K_i - K_{i-1})h_i}$ *is the optimal value to problem* (RJR).

The Lower Bound Theorem, given by Roundy [15] and extended by Queyranne [12] and Zheng [18] to general submodular setup cost functions (see also Federgruen et al. [6]) is a fundamental result for the analysis of heuristics for the joint replenishment problem.

Theorem 2.3 (Lower Bound Theorem). $\mathrm{LB}(n, K, h)$ *is a lower bound on the average cost of any feasible policy over any finite horizon.*

A *worst-case performance ratio* for a class of solutions to a problem is defined as the supremum, over all instances, of the ratio of the cost of the best policy in that class, to the optimum cost.

The following lemma is a direct extension of a rounding lemma in Roundy [15]. For $x \in \mathbb{R}_+$, let $e(x) \triangleq \frac{1}{2}(x + 1/x)$ denote the "EOQ sensitivity analysis function" and let $\log_\beta x$ denote the logarithm of x with base β.

Lemma 2.4 (Performance ratio $R_\beta(\beta_0)$ of power-of-β policies). *For any fixed base period β_0, let*

$$t_i^* \triangleq \beta_0 \beta^{\lfloor \log_\beta(t_i\sqrt{\beta}/\beta_0) \rfloor}, \quad \forall i \in N,$$

where $t = (t_1, t_2, \ldots, t_n)$ is an optimal solution to (RJR). *Then $t^* = (t_1^*, t_2^*, \ldots, t_n^*)$ is an optimal power-of-β policy for* (JR) *with β_0 fixed. The worst-case performance ratio $R_\beta(\beta_0)$ of power-of-β policies satisfies*

$$R_\beta(\beta_0) \triangleq \frac{C_\beta(n, K, h, \beta_0)}{\mathrm{LB}(n, K, h)} \le e(\sqrt{\beta}).$$

Proof. See Sun's thesis [17, pp. 171–172]. □

The next result, also a direct extension of Roundy, allows the base period β_0 to vary.

Lemma 2.5 (Performance ratio R_β^* of power-of-β policies). *Let $C_\beta(n, K, h) \triangleq \inf_{\beta_0 > 0} C_\beta(n, K, h, \beta_0)$ be the optimal average cost of power-of-β policies for instance (K, h) of n-product. The worst-case performance ratio R_β^* of power-of-β policies satisfies*

$$R_\beta^* \triangleq \inf_{\beta_0 > 0} R_\beta(\beta_0) = \frac{C_\beta(n, K, h)}{\mathrm{LB}(n, K, h)} \le \rho_\beta,$$

where $\rho_\beta \triangleq (1/\ln \beta)((\beta - 1)/\sqrt{\beta})$ is the upper bound of worst-case performance ratio of power-of-β policies.

Proof. See Sun's thesis [17, pp. 172–174]. □

3. Grouping policies and performance ratio

Say $G \triangleq \{G_1, G_2, \ldots, G_p\}$ is a *grouping* of N, if $\{G_1, G_2, \ldots, G_p\}$ forms a partition of N, i.e.,

$$\bigcup_{i=1}^{p} G_i = N; \qquad G_i \cap G_j = \emptyset \quad \text{iff} \quad i \neq j, \quad \forall i,j = 1, 2, \ldots, p.$$

Set G_i is called a *group* (of products). The corresponding *grouping policy* G will replenish all the products in each group G_i at the same time T_i, the *replenishment period* of group G_i. As we mentioned in the Introduction, we ignore the possible cost savings due to replenishing different groups at the same time. Therefore, the optimal average cost $g(G_i)$ of group G_i is

$$g(G_i) \triangleq 2\sqrt{K(G_i)h(G_i)},$$

the corresponding optimal replenishment period T_i^* of group G_i is

$$T_i^* = \sqrt{K(G_i)/h(G_i)},$$

and the optimal average cost $C_{\mathrm{GP}}(n, K, h, G)$ of grouping policy G is the sum of optimal average costs of all groups in G:

$$C_{\mathrm{GP}}(n, K, h, G) = \sum_{i=1}^{p} g(G_i),$$

where $|G| = p$, and the corresponding optimal replenishment period vector is $T^* = (T_1^*, T_2^*, \ldots, T_p^*)$.

If we use the (not necessarily optimal) replenishment period vector $T = (T_1, T_2, \ldots, T_p)$ for grouping policy G, the corresponding average cost is

$$C_{\mathrm{GP}}(n, K, h, G, T) \triangleq \sum_{i=1}^{p} \left\{ \frac{K(G_i)}{T_i} + h(G_i)\, T_i \right\}.$$

Therefore, the optimal average cost of grouping policy G can also be expressed as

$$C_{\mathrm{GP}}(n, K, h, G) = \inf_{T \in \mathbb{R}_+^{|G|}} C_{\mathrm{GP}}(n, K, h, G, T).$$

Let $\mathscr{G}^n$ be the set of all partitions on N, and

$$C_{\mathrm{GP}}(n, K, h, \mathscr{G}^n) \triangleq \min_{G \in \mathscr{G}^n} C_{\mathrm{GP}}(n, K, h, G)$$

be the average cost of *optimal grouping policies* for n-product instance (K, h).

Because we are mainly concerned with finding an upper bound on the performance ratio of grouping policies, we replace the optimal average cost by its lower bound $\mathrm{LB}(n, K, h)$ in our definition of performance ratio of grouping policies. This substitution cannot decrease the performance ratio. Therefore, the upper bound obtained here will also be an upper bound on the "real" performance ratio. However, because $\mathrm{LB}(n, K, h)$ is a very tight lower bound within 2% of the optimum cost, see Roundy [15] (by letting $\beta = 2$ in Lemma 2.5), we do not really lose much by this substitution.

Now we are ready to define various performance ratios of grouping policies. Let

$$r(n, K, h, G, T) \triangleq \frac{C_{\mathrm{GP}}(n, K, h, G, T)}{\mathrm{LB}(n, K, h)}$$

be the performance ratio of grouping policy G with replenishment period vector T for n-product instance (K, h). Let

$$r(n, K, h, G) \triangleq \frac{C_{\mathrm{GP}}(n, K, h, G)}{\mathrm{LB}(n, K, h)}$$

be the performance ratio of grouping policy G for n-product instance (K, h). Let

$$r(n, K, h, \mathscr{G}^n) \triangleq \frac{C_{\mathrm{GP}}(n, K, h, \mathscr{G}^n)}{\mathrm{LB}(n, K, h)}$$

be the performance ratio of an *optimal* grouping policy for n-product instance (K, h).

Let

$$r^*(n) \triangleq \sup_{(K,h)\in(\mathscr{S}^n,\mathbb{R}^n_+)} r(n,K,h,\mathscr{D}^n)$$

be the *worst-case* performance ratio of grouping heuristics for n-product, and let

$$r^* \triangleq \sup_{n\in\mathbb{N}} r^*(n)$$

be the *worst-case performance ratio of grouping heuristics* (over all problem instances). Note that $\mathbb{N} \triangleq \{1,2,3,\dots\}$ is the set of natural numbers. Estimating an upper bound on ratio r^* is the main objective of this paper.

A partition $G \in \mathscr{D}^n$ is a *consecutive partition* iff the indices of the products in each group G_i are consecutive integers. Let $\mathscr{C}^n$ be the set of all consecutive partitions on N. Lemmas 3.1–3.4 imply that, to find the worst-case performance ratio r^*, we may limit the problem instances to $(K,h) \in \Omega^n$ ("monotone path" instances), and grouping policies to $G \in \mathscr{C}^n$.

First, we may limit problem instances from submodular set functions ($\mathscr{S}^n$) to "maximum" submodular set functions ($\mathscr{M}^n$):

Lemma 3.1 (First reduction).

$$r^*(n) = \sup_{(K,h)\in(\mathscr{M}^n,\mathbb{R}^n_+)} r(n,K,h,\mathscr{D}^n).$$

Proof. See Appendix A. □

Observation. A by-product of the proof is that, given the solution of (RJR), a maximum submodular set function can be constructed from the submodular set function in linear time O(n). As we know from Queyranne [12] and Zheng [18] that the solution of (RJR) can be found in time polynomial in n, the maximum submodular function is also constructed in polynomial time.

Because of Lemma 3.1, we will assume $K \in \mathscr{M}^n$ hereafter. The following lemma shows that we may restrict attention to grouping policies using only consecutive partitions of N, which we call the *consecutive grouping heuristic*. Let

$$C_{\mathrm{GP}}(n,K,h,\mathscr{C}^n) \triangleq \min_{G\in\mathscr{C}^n} C_{\mathrm{GP}}(n,K,h,G)$$

be the average cost of *consecuive* grouping heuristics, for n-product instance (K,h). Let

$$r(n,,h,\mathscr{C}^n) \triangleq \frac{C_{\mathrm{GP}}(n,K,h,\mathscr{C}^n)}{\mathrm{LB}(n,K,h)}$$

be the corresponding performance ratio.

Lemma 3.2 (Consecutive grouping heuristics are optimal). *If* $K \in \mathscr{M}^n$, *then* $C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n) = C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n)$.

Proof. See Appendix A. □

Therefore, using the method proposed in Chakravarty, Orlin, and Rothblum [4], an optimal grouping policy can be found by an $\mathrm{O}(n^2)$ shortest path algorithm. Combining this with the observation following Lemma 3.1, we conclude that an optimal grouping policy can be found in polynomial time, roughly the same time as for finding an optimum power-of-two policy.

The second reduction follows directly from Lemmas 3.1 and 3.2:

Corollary 3.3 (Second reduction). *The worst-case performance ratio of consecutive grouping heuristics for all problem instances* $K \in \mathscr{M}^n$ *is* $r^*(n)$, *i.e.*,

$$r^*(n) = \sup_{(K,h) \in (\mathscr{M}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{C}^n).$$

The next lemma shows that we may further restrict our attention to problem instances $(K, h) \in \Omega^n$. Recall that Ω^n is the set of "monotone path" instances.

Lemma 3.4 (Third reduction). *The worst-case performance ratio of consecutive grouping heuristics for all problem instances* $(K, h) \in \Omega^n$ *is* $r^*(n)$, *i.e.*,

$$r^*(n) = \sup_{(K,h) \in \Omega^n} r(n, K, h, \mathscr{C}^n) = \sup_{(K,h) \in \Omega^n} \frac{\min_{G \in \mathscr{C}^n} 2\sum_{k=1}^{p} \sqrt{K_{l_k} h_{G_k}}}{2\sum_{i=1}^{n} \sqrt{(K_i - K_{i-1}) h_i}},$$

where

$$\left.\begin{aligned} G_k &= \{l_{k-1} + 1, \ldots, l_k\} \\ h_{G_k} &= \sum_{i \in G_k} h_i \end{aligned}\right\} \quad \text{for } k = 1, 2, \ldots, p.$$

Proof. See Appendix A. □

Now we only need to consider problem instances $(K, h) \in \Omega^n$. We find it convenient to replace variables (K, h) by (x, t) as follows:

$$\left.\begin{aligned} x_i &= \sqrt{(K_i - K_{i-1}) h_i}, \\ t_i &= \sqrt{(K_i - K_{i-1}) / h_i}, \end{aligned}\right\} \quad \forall\, i \in N. \tag{3.1}$$

Let $\mathbb{R}^n_< \triangleq \{t \in \mathbb{R}^n_+ \mid t_1 < t_2 < \cdots < t_n\}$ be the "monotone cone" in $\mathbb{R}^n_+$. It is easy to verify that $(K, h) \in \Omega^n$ is equivalent to $(x, t) \in (\mathbb{R}^n_+, \mathbb{R}^n_<)$. For simplicity, we do not change function names when variables (K, h) are replaced by (x, t).

The following lemma establishes the correspondence between variables (K, h) and (x, t).

Lemma 3.5 (Change of variables). *Let* $G = \{G_1, G_2, \ldots, G_p\} \in \mathscr{C}^n$, (x, t) *be defined by* (3.1), *and*

$$f_{jk}(t, T) \triangleq e\left(\frac{t_j}{T_k}\right) + \frac{1}{2}\sum_{i=k+1}^{p} \frac{t_j}{T_i}, \quad \forall j \in G_k,\ k = 1, 2, \ldots, p,$$

then for any $(K, h) \in \Omega^n$, *i.e.*, $(x, t) \in (\mathbb{R}^n_+, \mathbb{R}^n_<)$, *we have*

$$C_{\mathrm{GP}}(n, K, h, G, T) = 2\sum_{k=1}^{p}\sum_{j \in G_k} f_{jk}(t, T) x_j,$$

and

$$\mathrm{LB}(n, K, h) = 2\sum_{i=1}^{n} x_i.$$

Proof. Clearly,

$$h_i = x_i/t_i,\ K_i - K_{i-1} = x_i t_i, \quad \text{for } i = 1, 2, \ldots, n.$$

Therefore,

$$K_i = K_i - K_0 = \sum_{j=1}^{i} (K_j - K_{j-1}) = \sum_{j=1}^{i} x_j t_j, \quad \text{for } i = 1, 2, \ldots, n.$$

As $G = \{G_1, G_2, \ldots, G_p\} \in \mathscr{C}^n$, suppose

$$G_i = \{l_{i-1} + 1, \ldots, l_i\}, \quad \text{for } i = 1, 2, \ldots, p.$$

Then

$$K(G_i) = K_{l_i}, \quad h(G_i) = \sum_{j \in G_i} h_j, \quad \text{for } i = 1, 2, \ldots, p.$$

Therefore,

$$\begin{aligned}
C_{\mathrm{GP}}(n, K, h, G, T) &= \sum_{i=1}^{p}\left\{\frac{K(G_i)}{T_i} + h(G_i)\,T_i\right\} \\
&= \sum_{i=1}^{p}\left\{\frac{1}{T_i}\sum_{j=1}^{l_i} x_j t_j + T_i \sum_{j \in G_i} \frac{x_j}{t_j}\right\} \\
&= \sum_{i=1}^{p}\sum_{j=1}^{l_i} \frac{t_j}{T_i} x_j + \sum_{i=1}^{p}\sum_{j=l_{i-1}+1}^{l_i} \frac{T_i}{t_j} x_j \\
&= \sum_{k=1}^{p}\sum_{j=l_{k-1}+1}^{l_k}\sum_{i=k}^{p} \frac{t_j}{T_i} x_j + \sum_{k=1}^{p}\sum_{j=l_{k-1}+1}^{l_k} \frac{T_k}{t_j} x_j \\
&= \sum_{k=1}^{p}\sum_{j=l_{k-1}+1}^{l_k}\left\{\frac{t_j}{T_k} + \frac{T_k}{t_j} + \sum_{i=k+1}^{p} \frac{t_j}{T_i}\right\} x_j \\
&= 2\sum_{k=1}^{p}\sum_{j \in G_k} f_{jk}(t, T) x_j.
\end{aligned}$$

This completes the proof. $\square$

The following lemma gives an upper bound on the performance ratio of any grouping G.

Lemma 3.6 (An upper bound on the performance ratio of any grouping G). *Suppose $(K, h) \in \Omega^n$, or equivalently $(x, t) \in (\mathbb{R}^n_+, \mathbb{R}^n_<)$ defined by* (3.1). *For any grouping G, let*

$$W(n, t, G) \triangleq \inf_{T \in \mathbb{R}_+^{|G|}} \max_{\substack{j \in G_k \\ k = 1, 2, \ldots, |G|}} f_{jk}(t, T)$$

where $f_{jk}(t, T)$ is defined in Lemma 3.5. *Then we have*

$$\sup_{x \in \mathbb{R}^n_+} \inf_{T \in \mathbb{R}_+^{|G|}} \frac{C_{\mathrm{GP}}(n, K, h, G, T)}{\mathrm{LB}(n, K, h)} \leq W(n, t, G).$$

Proof. By definition, we have:

$$\begin{aligned} W(n, t, G) &= \inf_{T \in \mathbb{R}_+^{|G|}} \max_{\substack{j \in G_k \\ k = 1, 2, \ldots, |G|}} f_{jk}(t, T) \\ &\geq \inf_{T \in \mathbb{R}_+^{|G|}} \frac{\sum_{k=1}^{|G|} \sum_{j \in G_k} f_{jk}(t, T) x_j}{\sum_{j=1}^{n} x_j} \\ &= \inf_{T \in \mathbb{R}_+^{|G|}} \frac{C_{\mathrm{GP}}(n, K, h, G, T)}{\mathrm{LB}(n, K, h)}. \end{aligned}$$

This completes the proof. $\square$

4. Upper bound on the worst-case performance ratio of grouping policies

Based on the change of variables introduced in the previous section, we may rewrite the worst-case performance ratio of grouping heuristics $r^*(n)$ in the following form:

$$\begin{aligned} r^*(n) &= \sup_{(x,t) \in (\mathbb{R}^n_+, \mathbb{R}^n_<)} \frac{\min_{G \in \mathscr{C}^n} 2 \sum_{k=1}^{p} \sqrt{(\sum_{j=1}^{l_k} x_j t_j)(\sum_{j \in G_k} x_j / t_j)}}{2 \sum_{i=1}^{n} x_i} \\ &= \sup_{(x,t) \in (\mathbb{R}^n_+, \mathbb{R}^n_<)} \frac{\min_{G \in \mathscr{C}^n} \inf_{T \in R_+^{|G|}} 2 \sum_{k=1}^{p} \sum_{j \in G_k} f_{jk}(t, T) x_j}{2 \sum_{i=1}^{n} x_i}. \end{aligned}$$

This problem, with only $O(n)$ variables, is much simpler than the original one defined on submodular joint setup cost functions. An optimum consecutive grouping policy can be determined using an $O(n^2)$ shortest path algorithm [4]. However, finding an upper bound on this problem is still difficult for the following reasons:

(1) The objective function is neither quasiconcave nor quasiconvex in variables x and t. There are many local minima and maxima. Therefore, classic convex analysis methods do not apply directly.

(2) The problem is first to minimize on a set of consecutive grouping policies, then to maximize with respect to variables x and t. It cannot be solved by directly exchanging the min and sup operators.

(3) This problem also has many constraints.

Therefore, we estimate an upper bound r^* on the optimal value of this problem, instead of solving it directly. For this, we consider a subproblem, wherein we restrict attention to the set $\mathscr{B}_\beta^n$ of "root-of-β path" instances, defined as follows. Let

$$\mathscr{B}_\beta^n \triangleq \{(K,h) \in \Omega^n \mid m_1(\beta_0) < m_2(\beta_0) < \cdots < m_n(\beta_0) \text{ for some } \beta_0 \in [1/\sqrt{\beta}, \sqrt{\beta})\}$$

be the set of "root-of-β path" instances. Note that

$$m_i(\beta_0) \triangleq \left\lfloor \log_\beta\left(\sqrt{\frac{K_i - K_{i-1}}{h_i}}\,\frac{\sqrt{\beta}}{\beta_0}\right)\right\rfloor, \quad \forall i.$$

$$\mathbb{R}_\beta^n \triangleq \{t \in \mathbb{R}_<^n \mid m_1(\beta_0) < m_2(\beta_0) < \cdots < m_n(\beta_0) \text{ for some } \beta_0 \in [1/\sqrt{\beta}, \sqrt{\beta})\}$$

be the "root-of-β cone". Note that

$$m_i(\beta_0) \triangleq \left\lfloor \log_\beta\left(t_i \frac{\sqrt{\beta}}{\beta_0}\right)\right\rfloor, \quad \forall i.$$

It is clear that $(K,h) \in \mathscr{B}_\beta^n$ is equivalent to $(x,t) \in (\mathbb{R}_+^n, \mathbb{R}_\beta^n)$. Note that the strict inequalities in the definition imply that all the replenishment periods $t_i = \sqrt{(K_i - K_{i-1})/h_i}$ are in distinct "root-of-β" intervals $[1/\sqrt{\beta}, \sqrt{\beta})\beta_0\beta^{m_i}$. Therefore, we have $t_i/t_j > \beta^{m_i - m_j - 1} \geq \beta^{i-j-1}$, or equivalently $t_j/t_i < 1/\beta^{i-j-1} = \beta^{j-i+1}$, for all i, j, $i \geq j$. Although the worst-case performance ratio of grouping heuristics over the set $\mathscr{B}_\beta^n$ of "root-of-β path" instances need not be equal to the requisite ratio r^*, we show that the product of r_β^* and the over-all performance ratio ρ_β of power-of-β policies yields a valid upper bound: $r^* \leq \rho_\beta r_\beta^*$ for $\beta = 2, 3, \ldots$.

Theorem 4.1 (An upper bound from "root-of-β path" instances). *Let $r_\beta^* \triangleq \sup_{n\in\mathbb{N}} \sup_{(K,h)\in\mathscr{B}_\beta^n} r(n, K, h, \mathscr{C}^n)$ be the worst-case performance ratio of consecutive grouping heuristics over problem instances $(K,h) \in \mathscr{B}_\beta^n$, "root-of-$\beta$ path" instances, then the worst-case performance ratio of grouping heuristics r^* satisfies:*

$$r^* \leq \inf_{\beta \in \mathbb{N}\setminus\{1\}} \rho_\beta r_\beta^*.$$

Proof. See Appendix A. □

Since we know the value of ρ_β (see Lemma 2.5, and Table 1), we thus only need to find an upper bound W_β on r_β^*. For this, we now define a grouping heuristic H_b,

Input: Instance $(K, h) \in \mathscr{B}^n_\beta$, real numbers $\beta > 1$ and $b \in (1, \sqrt{\beta}]$.
Output: Number p of groups, and grouping $G(b) = \{G_1, G_2, \ldots, G_p\}$.
Step 1. **for** $i := 1$ **to** n **do** $t_i := \sqrt{(K_i - K_{i-1})/h_i}$;
$t_0 := 0$; $t_{n+1} := +\infty$;
Step 2. $i := 1$; $k := 0$;
repeat
$k := k + 1$;
if $t_{i+1} > bt_i$
then begin $G_k := \{i\}$; $i := i + 1$ **end**
else begin $G_k := \{i, i+1\}$; $i := i + 2$ **end**
until $i > n$;
$p := k$.

Fig. 1. Grouping heuristic H_b.

see Fig. 1, where $b \in \mathbb{R}_+$ is a parameter which satisfies $1 < b \leq \sqrt{\beta}$ and whose precise value will be determined later. This heuristic assigns at most two products to every group.

Note that the values assigned in this heuristic H_b to the vector t are consistent with the change of variables (3.1). The following lemma relates these values to that of parameter b. As such group G_k $(k = 1, 2, \ldots, p)$ in the grouping produced by heuristic H_b contains at most two products, let $l_k \triangleq \max\{i \mid i \in G_k\}$, so that $G_k = \{l_k\}$ or $G_k = \{l_k - 1, l_k\}$.

Lemma 4.2. *Suppose t is the vector generated by heuristic H_b. Then we have $t_{l_k+1} > bt_{l_k}$ for all k.*

Proof. If $G_k = \{i\}$, then $t_{l_k+1} = t_{i+1} > bt_i = bt_{l_k}$. Otherwise, $G_k = \{i, i+1\}$, and using $t_{i+2} > \beta t_i$ and $b \leq \sqrt{\beta}$, we have $t_{l_k+1} = t_{i+2} > \beta t_i \geq \beta(t_{i+1}/b) \geq bt_{i+1} = bt_{l_k}$. $\square$

The following technical lemma is used in the proof of Lemma 4.4:

Lemma 4.3. *If $1 \leq y \leq b$ and $0 < a_2 \leq \sqrt{b}$, then we have*

$$e\left(\frac{y}{a_2}\right) \leq e\left(\frac{b}{a_2}\right).$$

Proof. See Appendix A. $\square$

To get an upper bound on $f_{jk}(t, T)$, we introduce two additional parameters a_1 and a_2 satisfying $0 < a_1$ and $0 < a_2 \leq \sqrt{b}$, and which will be optimized later. The following lemma provides six inequalities involving these two parameters, and which we use to simplify the estimation of an upper bound on W_β.

For any grouping policy $G(b) = \{G_1, G_2, \ldots, G_p\}$ with $|G_i| \in \{1, 2\}$ for all $i = 1, \ldots, p$, generated by heuristic H_b, let

$$T_k \triangleq t_{l_k}/a_{|G_k|}, \quad \text{for } k = 1, 2, \ldots, p, \tag{4.1}$$

$$f^b_{jk}(t, T) \triangleq e\left(a_{|G_k|}\frac{t_j}{t_{l_k}}\right) + \frac{t_j}{2}\sum_{i=k+1}^{p}\frac{a_{|G_i|}}{t_{l_i}}. \tag{4.2}$$

Lemma 4.4 (Upper bound on $f^b_{jk}(t, T)$). *For any $\beta > 1$ and b satisfying $1 < b \le \sqrt{\beta}$, let*:

$$c \triangleq \max\left(\frac{a_1}{\beta}, \frac{a_2}{\beta^2}\right),$$

$$g_1(a_1, a_2, b) \triangleq \frac{1}{2}\left(a_1 + \frac{1}{a_1} + \frac{a_1}{b} + \frac{c\beta}{\beta - 1}\right),$$

$$g_2(a_1, a_2, b) \triangleq \frac{1}{2}\left(a_1 + \frac{1}{a_1} + \frac{a_2}{\beta} + \frac{c}{\beta - 1}\right),$$

$$g_3(a_1, a_2, b) \triangleq \frac{1}{2}\left(a_2 + \frac{1}{a_2} + a_1\frac{b}{\beta} + \frac{c\beta}{\beta - 1}\right),$$

$$g_4(a_1, a_2, b) \triangleq \frac{1}{2}\left(a_2 + \frac{1}{a_2} + \frac{a_2}{\beta} + \frac{c}{\beta - 1}\right),$$

$$g_5(a_1, a_2, b) \triangleq \frac{1}{2}\left(\frac{a_2}{b} + \frac{b}{a_2} + \frac{a_1}{\beta} + \frac{c}{\beta - 1}\right),$$

$$g_6(a_1, a_2, b) \triangleq \frac{1}{2}\left(\frac{a_2}{b} + \frac{b}{a_2} + \frac{a_2}{\beta^2} + \frac{c}{\beta(\beta - 1)}\right).$$

Then, for $t \in \mathbb{R}^n_\beta$ and any a_1, a_2 satisfying $0 < a_1$ and $0 < a_2 \le \sqrt{b}$, we have, for $j = 1, 2, \ldots, n$:

$$f^b_{jk}(t, T) \le \max_{i=1,\ldots,6} g_i(a_1, a_2, b),$$

where T and $f^b_{jk}(t, T)$ are defined by equation (4.1) *and* (4.2) *respectively.*

Proof. Observe that

$$l_i - l_k = \sum_{r=k+1}^{i} |G_r|, \quad \text{for } i = k + 1, \ldots, p.$$

As $t \in \mathbb{R}^n_\beta$, we have

$$\frac{t_i}{t_j} > \beta^{i-j-1}, \quad \forall i,j, i \geq j,$$

$$\frac{t_{l_i}}{t_j} \leq \frac{1}{\beta^{l_i - j - 1}}, \quad \forall j > l_i.$$

Therefore, we have

$$t_j \sum_{i=k+2}^{p} \frac{a_{|G_i|}}{t_{l_i}} \leq \sum_{i=k+2}^{p} \frac{a_{|G_i|}}{\beta^{l_i - j - 1}},$$

and

$$\begin{aligned} t_{l_k} \sum_{i=k+2}^{p} \frac{a_{|G_i|}}{t_{l_i}} &\leq \sum_{i=k+2}^{p} \frac{a_{|G_i|}}{\beta^{l_i - l_k - 1}} \\ &= \frac{1}{\beta^{|G_{k+1}|-1}} \left(\frac{a_{|G_{k+2}|}}{\beta^{|G_{k+2}|}} + \frac{a_{|G_{k+3}|}}{\beta^{|G_{k+2}|+|G_{k+3}|}} \right. \\ &\qquad \left. + \cdots + \frac{a_{|G_p|}}{\beta^{|G_{k+2}|+\cdots+|G_p|}} \right) \\ &\leq \frac{c}{\beta^{|G_{k+1}|-1}} \left(1 + \frac{1}{\beta^{|G_{k+2}|}} + \cdots + \frac{1}{\beta^{|G_{k+2}|+\cdots+|G_{p-1}|}} \right) \\ &\leq \frac{c}{\beta^{|G_{k+1}|-1}} \frac{1}{1 - 1/\beta} \\ &= \frac{c}{\beta^{|G_{k+1}|-1}} \frac{\beta}{\beta - 1}. \end{aligned}$$

There are two cases to consider:

Case 1: $G_k = \{l_k\}$, $j = l_k$. Using Lemma 4.2, we have

$$t_{l_k} \frac{a_{|G_{k+1}|}}{t_{l_{k+1}}} \leq \begin{cases} \dfrac{a_1}{b}, & \text{if } |G_{k+1}| = 1, \\ \dfrac{a_2}{\beta}, & \text{if } |G_{k+1}| = 2. \end{cases}$$

Therefore,

$$\begin{aligned} f^b_{l_k,k}(t, T) &= e(a_1) + \frac{t_{l_k}}{2} \sum_{i=k+1}^{p} \frac{a_{|G_i|}}{t_{l_i}} \\ &\leq \begin{cases} \dfrac{1}{2}\left(a_1 + \dfrac{1}{a_1} + \dfrac{a_1}{b} + \dfrac{c\beta}{\beta - 1} \right), & \text{if } |G_{k+1}| = 1, \\ \dfrac{1}{2}\left(a_1 + \dfrac{1}{a_2} + \dfrac{a_2}{\beta} + \dfrac{c}{\beta - 1} \right), & \text{if } |G_{k+1}| = 2. \end{cases} \end{aligned}$$

Case 2: $G_k = \{l_k - 1, l_k\}$. Observe that $1 < t_{l_k}/t_{l_k-1} \le b$, and $t_{l_k+1}/t_{l_k} > \beta/b \ge b$. We have two subcases:

Case 2.1: $j = l_k$.

$$f^b_{l_k,k}(t,T) = e(a_2) + \frac{t_{l_k}}{2}\sum_{i=k+1}^{p}\frac{a_{|G_i|}}{t_{l_i}}$$

$$\le \begin{cases} \frac{1}{2}\left(a_2 + \frac{1}{a_2} + a_1\frac{b}{\beta} + \frac{c\beta}{\beta-1}\right), & \text{if } |G_{k+1}| = 1,\\ \frac{1}{2}\left(a_2 + \frac{1}{a_2} + \frac{a_2}{\beta} + \frac{c}{\beta-1}\right), & \text{if } |G_{k+1}| = 2.\end{cases}$$

Case 2.2: $j = l_k - 1$. Because $1 \le t_{l_k}/t_{l_k-1} \le b$ and $0 < a_2 \le \sqrt{b}$, by the inequality in Lemma 4.3, we have $e((t_{l_k}/t_{l_k-1})/a_2) \le e(b/a_2)$. Because $t \in \mathbb{R}^n_\beta$, we have

$$t_{l_k-1}\sum_{i=k+2}^{p}\frac{a_{|G_i|}}{t_{l_i}} \le \sum_{i=k+2}^{p}\frac{a_{|G_i|}}{\beta^{l_i-l_k}} \le \frac{c}{\beta^{|G_{k+1}|}}\frac{\beta}{\beta-1}.$$

Therefore

$$f^b_{l_k-1,k}(t,T) = e\left(a_2\frac{t_{l_k-1}}{t_{l_k}}\right) + \frac{t_{l_k-1}}{2}\sum_{i=k+1}^{p}\frac{a_{|G_i|}}{t_{l_i}}$$

$$\le \begin{cases} \frac{1}{2}\left(\frac{b}{a_2} + \frac{a_2}{b} + \frac{a_1}{\beta} + \frac{c}{\beta-1}\right), & \text{if } |G_{k+1}| = 1,\\ \frac{1}{2}\left(\frac{b}{a_2} + \frac{a_2}{b} + \frac{a_2}{\beta^2} + \frac{c}{\beta(\beta-1)}\right), & \text{if } |G_{k+1}| = 2.\end{cases}$$

This completes our proof. □

Corollary 4.5 (Upper bound on $W(n,t,G(b))$). *Suppose $G(b)$ is the grouping generated by heuristic H_b, and let*

$$W_\beta \triangleq \min_{a_1,a_2,b}\left\{\max_{i=1,\dots,6} g_i(a_1,a_2,b)\,|\,0 < a_1,\ 0 < a_2 \le \sqrt{b},\ 1 < b \le \sqrt{\beta}\right\}.$$

Then

$$W(n,t,G(b)) \le W_\beta.$$

We are now in a position to prove the main result of this paper:

Theorem 4.6 (Upper bound on the performance ratio of grouping policies). *The following is an upper bound on the average cost of a worst-case performance ratio of grouping policies:*

$$r^* \le \inf_{\beta\in\mathbb{N}\setminus\{1\}} \rho_\beta W_\beta \le 1.4480.$$

Proof. For any $\beta \in \mathbb{N}\backslash\{1\}$, by definition of r_β^* in Theorem 4.1, we have

$$
\begin{aligned}
r_\beta^* &= \sup_{n \in \mathbb{N}} \sup_{(K,h) \in \mathscr{B}_\beta^n} r(n, K, h, \mathscr{C}^n) \\
&= \sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}_\beta^n} \sup_{x \in \mathbb{R}_+^n} \frac{\min_{G \in \mathscr{C}^n} \inf_{T \in \mathbb{R}_+^{|G|}} C_{\mathrm{GP}}(n, K, h, G, T)}{\mathrm{LB}(n, K, h)} \\
&\le \sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}_\beta^n} \sup_{x \in \mathbb{R}_+^n} \inf_{T \in \mathbb{R}_+^{|G|}} \frac{C_{\mathrm{GP}}(n, K, h, G(b), T)}{\mathrm{LB}(n, K, h)} \\
&\le \sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}_\beta^n} W(n, t, G(b)) \quad \text{(by Lemma 3.6)} \\
&\le W_\beta \quad \text{(by Corollary 4.5).}
\end{aligned}
$$

Therefore, by Theorem 4.1, we have

$$
r^* \le \inf_{\beta \in \mathbb{N} \backslash \{1\}} \rho_\beta W_\beta .
$$

In Table 1, we list the values of ρ_β and upper bounds on W_β and r^*, for $\beta = 2, \ldots, 9$. From this table we get the best estimate from $\beta = 5$. More specifically, let $\beta = 5$, $\rho_\beta = 1.111478$, $a_1 = 0.7675$, $a_2 = 1.2243$, $b = 2.2361$. We have $g_1(a_1, a_2, b) = 1.3028$, $g_2(a_1, a_2, b) = 1.1768$, $g_3(a_1, a_2, b) = 1.2881$, $g_4(a_1, a_2, b) = 1.1622$, $g_5(a_1, a_2, b) = 1.2829$, $g_6(a_1, a_2, b) = 1.2153$. Therefore, $W_\beta \le 1.3028$, and $\rho_\beta W_\beta \le 1.4480$. □

The following example appeared in Zheng [18], and was independently found by the authors: $K_0 = 0$, $K_i = 3^i$, $h_i = 3^{-i}$, for $i = 1, 2, \ldots, n$, which produces a lower bound $n/(1 + (n-1)\sqrt{2/3})$ on $r^*(n)$. When n goes to infinity, the limit is $\sqrt{2/3}$.

Table 1
Performance ratios of power-of-β policies and estimates of W_β and r^*

	Base period			
	Fixed	Variable		
β	$\frac{1}{2}(\sqrt{\beta} + 1/\sqrt{\beta})$	$\rho_\beta = (1/\ln \beta)(\beta - 1)/\sqrt{\beta}$	W_β	$r^* \le \rho_\beta W_\beta$
2	1.06066	$\frac{1}{\sqrt{2}\ln 2} = 1.02014$	≤ 1.6453	≤ 1.6785
3	1.15470	$\frac{2}{\sqrt{3}\ln 3} = 1.05105$	≤ 1.4413	≤ 1.5149
4	1.25000	$\frac{3}{4\ln 2} = 1.08202$	≤ 1.3540	≤ 1.4651
5	1.34164	$\frac{4}{\sqrt{5}\ln 5} = 1.11148$	≤ 1.3028	≤ 1.4480
6	1.42887	$\frac{5}{\sqrt{6}\ln 6} = 1.13924$	≤ 1.2730	≤ 1.4503
7	1.51186	$\frac{6}{\sqrt{7}\ln 7} = 1.16541$	≤ 1.2625	≤ 1.4714
8	1.59099	$\frac{7}{\sqrt{8}\ln 8} = 1.19016$	≤ 1.2529	≤ 1.4912
9	1.66667	$\frac{4}{3\ln 3} = 1.21365$	≤ 1.2571	≤ 1.5257

Therefore, the bounds on r^* we have established herein are:

$$\sqrt{2/3} = 1.2247 \leq r^* \leq 1.4480.$$

5. Conclusion

Grouping policies have been widely used for their simplicity of implementation. If the setup cost function is separable, i.e., if there is no saving on joint replenishment, then a particular grouping policy—the EOQ solution for each product—is optimal and thus generally outperforms power-of-two policies. But when the setup cost function is *not* separable, the average cost of a best power-of-two policy can be as much as 20% below that of a best grouping policy, as shown by the above example. For related inventory models with network structure, some apparently reasonable heuristics, such as nested policies, integer-multiple lot size policies and integer-split lot size policies can be arbitrarily bad, see Roundy [15] and Atkins et al. [3]. It is thus of interest to determine whether or not grouping policies can also be arbitrarily bad for the joint replenishment problem. As the submodular setup joint cost is fairly general and allows the derivation of very tight bounds on the optimal cost, we have limited ourselves to this case.

In this paper we used a novel approach to obtain an upper bound on the performance ratio of grouping heuristics. We found that this performance ratio is finite and no greater than 1.448, i.e., the average cost of a best grouping policy is within 44.8% of the optimum. We believe that, not only this result, but also the method used to derive it, are of interest, and may apply to the analysis of a broad class of related inventory problems.

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Appendix A: The proofs of lemmas in the text

Lemma 3.1 (First reduction).

$$r^*(n) = \sup_{(K,h)\in(\mathscr{S}^n,\mathbb{R}^n_+)} r(n, K, h, \mathscr{D}^n).$$

Proof. By Theorem 2.1, for any $K \in \mathscr{S}^n$, there exist q distinct values $t(1) < t(2) < \cdots < t(q)$, and a nested path $(S_1, S_2, \ldots, S_q)$ in N with $S_l \setminus S_{l-1} = \{i \in N \mid t_i = t(l)\}$,

$t(l) = \sqrt{(K(S_l) - K(S_{l-1}))/h(S_l \backslash S_{l-1})}$, for $l = 1, 2, \ldots, p$, such that

$$\mathrm{LB}(n, K, h) = 2 \sum_{l=1}^{q} \sqrt{[K(S_l) - K(S_{l-1})] h(S_l \backslash S_{l-1})},$$

where $S_0 = \emptyset$. Now define a new set function $K': 2^N \mapsto \mathbb{R}_+$ such that for all $S \subseteq N$,

$$K'(S) \triangleq \min\{K(S_l) \mid S \subseteq S_l, \text{ for } l = 1, 2, \ldots, q\}.$$

It is obvious that $K'(S)$ is nondecreasing, that is, if $S \subseteq T$, then $K'(S) \leq K'(T)$. Because $K(S)$ is nondecreasing, $K'(S) \geq K(S)$, for all $S \subseteq N$. Therefore, we have $C_{\mathrm{GP}}(n, K', h, \mathscr{D}^n) \geq C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n)$.

It is easy to verify that all the conditions of Theorem 2.1 hold for problem instance (K', h). Therefore, by $K'(S_l) = K(S_l)$, for $l = 1, 2, \ldots, q$, we have

$$\mathrm{LB}(n, K', h) = 2 \sum_{l=1}^{q} \sqrt{[K'(S_l) - K'(S_{l-1})] h(S_l \backslash S_{l-1})} = \mathrm{LB}(n, K, h).$$

Observe that $\emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_q = N$ and $0 = K(S_0) < K(S_1) \leq K(S_2) \leq \cdots \leq K(S_q) = K(N)$. If $i \in S_j \backslash S_{j-1}$, then, since K is nondecreasing,

$$\begin{aligned} K'(\{i\}) &= \min\{K(S_l) \mid \{i\} \subseteq S_l, \text{ for } l = 1, 2, \ldots, q\} \\ &= \min_{l \in \{j, j+1, \ldots, q\}} K(S_l) = K(S_j). \end{aligned} \tag{A.1}$$

For any $S \subseteq N$, let $j = \min\{l \mid S \subseteq S_l\}$, then there exists $i \in S \cap (S_j \backslash S_{j-1})$. Therefore, by definition and equation (A.1), $K'(S) = K(S_j) = K'(\{i\})$. However, $\{i\} \in S$ implies that $K'(\{i\}) \leq \max_{k \in S} K'(\{k\}) \leq K'(S)$. Therefore, we have $K'(S) = \max_{k \in S} K'(\{k\})$, for all $S \subseteq N$.

Now let $K'_i \triangleq K'(\{i\})$, and we reorder the index set N such that $0 < K'_1 \leq K'_2 \leq \cdots \leq K'_n$, we have $K' \in \mathscr{M}^n$.

Therefore, for each $K \in \mathscr{S}^n$, there exists a $K' \in \mathscr{M}^n$ such that

$$\begin{aligned} r(n, K', h, \mathscr{D}^n) &= \frac{C_{\mathrm{GP}}(n, K', j, \mathscr{D}^n)}{\mathrm{LB}(n, K', h)} \\ &\geq \frac{C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n)}{\mathrm{LB}(n, K', h)} = \frac{C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n)}{\mathrm{LB}(n, K, h)} \\ &= r(n, K, h, \mathscr{D}^n), \end{aligned}$$

a nd we get

$$\sup_{(K,h) \in (\mathscr{M}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{D}^n) \geq \sup_{(K,h) \in (\mathscr{S}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{D}^n).$$

However, $\mathscr{M}^n \subset \mathscr{S}^n$ implies the reverse inequality. Hence, we have

$$\sup_{(K,h) \in (\mathscr{M}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{D}^n) = \sup_{(K,h) \in (\mathscr{S}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{D}^n).$$

This completes the proof. □

The following claim is useful in proving Lemma 3.2: consecutive grouping heuristics are optimal.

Claim (Consecutive grouping of three products). *If* $\xi_3 \geq \xi_2 > 0$, $\eta_1, \eta_2, \eta_3 > 0$, *then*

$$\sqrt{\xi_2 \eta_2} + \sqrt{\xi_3(\eta_1 + \eta_3)} > \min\{\sqrt{\xi_2(\eta_1 + \eta_2)} + \sqrt{\xi_3 \eta_3}, \sqrt{\xi_3(\eta_1 + \eta_2 + \eta_3)}\}.$$

Proof. Let

$$f(x) = \sqrt{\xi_2 x} + \sqrt{\xi_3(\eta_1 + \eta_2 + \eta_3 - x)}.$$

The inequality states that

$$f(\eta_2) \geq \min\{f(0), f(\eta_1 + \eta_2)\}.$$

Since $f(x)$ is strictly concave for $x \in [0\cdot, \eta_1 + \eta_2 + \eta_3]$, the result follows from the fact that $\eta_2 \in (0, \eta_1 + \eta_2 + \eta_3)$. □

Lemma 3.2 (Consecutive grouping heuristics are optimal). *If* $(K, h) \in (\mathscr{M}^n, \mathbb{R}^n_+)$, *then* $C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n) = C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n)$.

Proof. Because of $\mathscr{C}^n \subset \mathscr{D}^n$, we have

$$C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n) \geq C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n).$$

For the converse inequality, we show that, if $(K, h) \in (\mathscr{M}^n, \mathbb{R}^n_+)$, and partition $G = (G_1, G_2, \ldots, G_p) \in \mathscr{D}^n \backslash \mathscr{C}^n$, then

$$C_{\mathrm{GP}}(n, K, h, G) > C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n). \tag{A.2}$$

We prove (A.2) above by induction on n: Let $\xi_3 = K_3$, $\xi_2 = K_2$, $\eta_1 = h_1$, $\eta_2 = h_2$, $\eta_3 = h_3$. Then inequality (A.2) follows from inequality (A.2) for the case of $n = 3$. Next, suppose inequality (A.2) holds for n in general.

For any instance $(K, h) \in (\mathscr{M}^{n+1}, \mathbb{R}^{n+1}_+)$, let $G = (G_1, G_2, \ldots, G_p)$ be a partition of $N_1 \triangleq \{1, 2, \ldots, n, n+1\}$ such that $G \notin \mathscr{C}^{n+1}$. To show that inequality (A.2) also holds, we consider two cases:

Case 1. If there is at least one group in G containing two consecutive indices $j, j+1$, then we define $(K', h') \in (\mathscr{M}^n, \mathbb{R}^n_+)$ by combining j and $j+1$ together, i.e., let

$$K'_i = K_i, \qquad h'_i = h_i, \qquad \text{for } i = 1, 2, \ldots, j-1;$$

$$K'_j = K_{j+1}, \qquad h'_j = h_j + h_{j+1};$$

$$K'_i = K_{i+1}, \qquad h'_i = h_{i+1}, \qquad \text{for } i = j+1, j+2, \ldots, n.$$

We also define partition $G' \triangleq (G'_1, G'_2, \ldots, G'_p)$ of N by letting, for $i = 1, 2, \ldots, p$,

$$G'_i \triangleq \{m \mid m \in G_i, m \leq j\} \cup \{m \mid m + 1 \in G_i, m > j + 1\}.$$

Every grouping of n-product for (K', h') induces a grouping of $(n+1)$-product for (K, h) with the same cost. Therefore,

$$C_{\mathrm{GP}}(n+1, K, h, \mathscr{D}^{n+1}) \leq C_{\mathrm{GP}}(n, K', h', \mathscr{D}^n).$$

It is obvious that

$$C_{\mathrm{GP}}(n+1, K, h, G) = C_{\mathrm{GP}}(n, K', h', G').$$

Note that $G \notin \mathscr{C}^{n+1}$ implies $G' \notin \mathscr{C}^n$, and $(K', h') \in (\mathscr{M}^n, \mathbb{R}^n_+)$, by the induction we have:

$$C_{\mathrm{GP}}(n, K', h', G') > C_{\mathrm{GP}}(n, K', h', \mathscr{D}^n).$$

The last three inequalities imply inequality (A.2) for $n+1$.

Case 2. If there is no group in G which contains consecutive indices, let group G_p contain product $n+1$. There are two subcases to consider:

Case 2.1. Group G_p contains only one product, i.e., $G_p = \{n+1\}$. Since partition $G \notin \mathscr{C}^{n+1}$, partition $G' \triangleq (G_1, \ldots, G_{p-1}) \notin \mathscr{C}^n$. Let (K', h') denote the restriction of (K, h) to production set $N = \{1, 2, \ldots, n\}$. By induction, we have $C_{\mathrm{GP}}(n, K', h', G') > C_{\mathrm{GP}}(n, K', h', \mathscr{D}^n)$. Note that $C_{\mathrm{GP}}(n+1, K, h, G) = 2\sqrt{K_{n+1} h_{n+1}} + C_{\mathrm{GP}}(n, K', h', G')$, and $C_{\mathrm{GP}}(n, K', h', \mathscr{D}^n) + 2\sqrt{K_{n+1} h_{n+1}} \geq C_{\mathrm{GP}}(n+1, K, h, \mathscr{D}^{n+1})$. We get that inequality (A.2) holds for $n+1$.

Case 2.2. Group G_p contains more than one product, but does not contain product n. (Otherwise, Case 1 applies.) Let $G'_p = G_p \backslash \{n+1\}$: we have $G'_p \neq \emptyset$, and $n \notin G'_p$. W.l.o.g, suppose $n \in G_{p-1}$. To apply inequality (A.2), let $\xi_3 \triangleq K_{n+1} = K(G_p)$, $\xi_2 \triangleq K_n = K(G_{p-1}) = K(G_{p-1} \cup G'_p)$, $\eta_1 \triangleq h(G'_p) > 0$, $\eta_2 \triangleq h(G_{p-1}) > 0$, $\eta_3 \triangleq h(G_p) - h(G'_p) > 0$. It is easy to verify that the conditions of inequality (A.2) hold. Therefore,

$$\begin{aligned} &\sqrt{K_n h(G_{p-1})} + \sqrt{K_{n+1} h(G_p)} \\ &\quad > \min\left\{\sqrt{K_n h(G'_p \cup G_{p-1})} + \sqrt{K_{n+1} h_{n+1}}, \sqrt{K_{n+1} h(G_p \cup G_{p-1})}\right\}. \end{aligned}$$

Let

$$\begin{aligned} G' &= (G_1, G_2, \ldots, G_{p-2}, G_{p-1} \cup G'_p, \{n+1\}), \\ G'' &= (G_1, G_2, \ldots, G_{p-2}, G_{p-1} \cup G_p). \end{aligned}$$

Then we have

$$C_{\mathrm{GP}}(n+1, K, h, G) > \min\{C_{\mathrm{GP}}(n+1, K, h, G'), C_{\mathrm{GP}}(n+1, K, h, G'')\}.$$

As G' fits Case 2.1, we have

$$C_{\mathrm{GP}}(n+1, K, h, G') > C_{\mathrm{GP}}(n+1, K, h, \mathscr{D}^n).$$

As G'' fits Case 1, we have

$$C_{\mathrm{GP}}(n+1, K, h, G'') > C_{\mathrm{GP}}(n+1, K, h, \mathscr{D}^n).$$

The last three inequalities imply inequality (A.2) for $n+1$, and this completes the proof. □

Lemma 3.4 (Third reduction). *The worst-case performance ratio of consecutive grouping heuristics for all problem instances* $(K, h) \in \Omega^n$ *is* $r^*(n)$, *i.e.*,

$$r^*(n) = \sup_{(K,h)\in\Omega^n} r(n, K, h, \mathscr{C}^n)$$

$$= \sup_{(K,h)\in\Omega^n} \frac{\min_{G\in\mathscr{C}^n} 2\sum_{k=1}^{p} \sqrt{K_{l_k} h_{G_k}}}{2\sum_{i=1}^{n} \sqrt{(K_i - K_{i-1})h_i}},$$

where

$$\left.\begin{aligned} G_k &= \{l_{k-1}+1, \dots, l_k\} \\ h_{G_k} &= \sum_{i\in G_k} h_i \end{aligned}\right\} \text{ for } p = 1, 2, \dots, p.$$

Proof. The inequality $\sup_{(K,h)\in\Omega^n} r(n, K, h, \mathscr{C}^n) \le r^*(n)$ follows from $\Omega^n \subseteq (\mathscr{M}^n, \mathbb{R}^n_+)$. For the converse inequality, suppose $(K, h) \in (\mathscr{M}^n, \mathbb{R}^n_+)$. By Theorem 2.1, there exists a nested path $(S_1, S_2, \dots, S_q)$ such that

$$\mathrm{LB}(n, K, h) = 2 \sum_{l=1}^{q} \sqrt{[K(S_l) - K(S_{l-1})]h(S_l \backslash S_{l-1})}$$

and

$$t(l) = \sqrt{[K(S_l) - K(S_{l-1})]/h(S_l \backslash S_{l-1})}, \quad \text{with } t(1) < t(2) < \cdots < t(q).$$

Define a q-product problem instance (q, K', h') by

$$\left.\begin{aligned} K'_l &\triangleq K(S_l) \\ K'(S) &\triangleq \max_{l\in S} K'_l \\ h'_l &\triangleq h(S_l \backslash S_{l-1}) \end{aligned}\right\} \text{ for } l = 1, 2, \dots, q, \text{ and } \forall S \subseteq \{1, 2, \dots, q\}.$$

It is obvious that $(K', h') \in \Omega^q$.

By inequality (A.2), we have

$$\mathrm{LB}(q, K', h') = 2 \sum_{l=1}^{q} \sqrt{[K'_l - K'_{l-1}]h'_l} = \mathrm{LB}(n, K, h).$$

Because each grouping policy in $\mathscr{C}^q$ with problem instance (K', h') corresponds to a grouping in $\mathscr{C}^n$ with problem instance (K, h) having the same average cost, we have

$$C_{\mathrm{GP}}(q, K', h', \mathscr{C}^q) \ge C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n).$$

Therefore, $r(q, K', h', \mathscr{C}^q) \ge r(n, K, h, \mathscr{C}^n)$, that is, for any $(K, h) \in (\mathscr{M}^n, \mathbb{R}^n_+)$, we have

$$\sup_{q\le n} \sup_{(K'',h'')\in\Omega^n} r(q, K'', h'', \mathscr{C}^q) \ge r(n, K, h, \mathscr{C}^n),$$

which implies the requisite reverse inequality, and completes the proof. $\square$

Theorem 4.1 (An upper bound from "root-of-β path" instances). *Let* $r^*_\beta \triangleq \sup_{n\in\mathbb{N}} \sup_{(K,h)\in\mathscr{B}^n_\beta} r(n,K,h,\mathscr{C}^n)$ *be the performance ratio of consecutive grouping policy for the "root-of-β path" instances, then the worst-case performance ratio of grouping policies* r^* *satisfies:*

$$r^* \le \inf_{\beta\in\mathbb{N}\setminus\{1\}} \rho_\beta r^*_\beta.$$

Proof. By Lemma 2.5, for any $(K,h)\in(\mathscr{M}^n,\mathbb{R}^n_+)$, there exist $\beta_0\in[1/\sqrt{\beta},\sqrt{\beta})$ and a power-of-β policy associated with a nested path $(S_1,S_2,\dots,S_q)$ with base period β_0, such that

$$C_\beta(n,K,h,\beta_0) \le \rho_\beta\,\mathrm{LB}(n,K,h),$$

where

$$C_\beta(n,K,h,\beta_0) = \sum_{l=1}^{q}\left\{\frac{K(S_l)-K(S_{l-1})}{t^*(l)} + h(S_l\setminus S_{l-1})t^*(l)\right\},$$

$$\mathrm{LB}(n,K,h) = 2\sum_{l=1}^{q}\sqrt{[K(S_l)-K(S_{l-1})]h(S_l\setminus S_{l-1})},$$

for $l=1,2,\dots,q$:

$$t(l) = \sqrt{\frac{K(S_l)-K(S_{l-1})}{h(S_l\setminus S_{l-1})}} \in [1/\sqrt{\beta},\sqrt{\beta})t^*(l),$$

$$t^*(l) = \beta_0\beta^{n_l},\ n_l\in\mathbb{Z},\quad \text{with } n_1\le n_2\le\cdots\le n_q,$$

$$t^*_i = t^*(l),\quad \text{if } i\in S_l\setminus S_{l-1}.$$

Note that the consecutive inequalities between $\{n_1,n_2,\dots,n_q\}$ are not strict. In the following we define a problem instance $(K',h')\in\mathscr{B}^j_\beta$ by putting all products with the same replenishment period t^*_i into a same group. Suppose $\{n_1,n_2,\dots,n_q\}$ take j different values, i.e., let $(r_1,r_2,\dots,r_j)$ satisfy

$$n_{r_{k-1}+1} = \cdots = n_{r_k},\quad \text{for } k=1,2,\dots,j,$$

$$n_{r_k} < n_{r_k+1},\qquad \text{for } k=1,2,\dots,j-1,$$

where $r_0=0$, $r_j=q$. We recursively define S'_k by letting

$$S'_0 \triangleq \emptyset,$$

$$S'_k\setminus S'_{k-1} \triangleq S_{r_k}\setminus S_{r_{k-1}+1}\quad \text{for } k=1,2,\dots,j.$$

Partition $\{S'_1,S'_2,\dots,S'_j\}$ has the following properties:

(1) $\{S'_1,S'_2,\dots,S'_j\}$ forms a nested path.

(2) $t^*_i = t^*(r_k) = \beta_0\beta^{n_{r_k}}$, if $i\in S'_k\setminus S'_{k-1}$.

(3) Note also that

$$S'_k \backslash S'_{k-1} = \bigcup_{l=r_{k-1}+1}^{r_k} (S_l \backslash S_{l-1}),$$

$$K(S'_k) - K(S'_{k-1}) = \sum_{l=r_{k-1}+1}^{r_k} [K(S_l) - K(S_{l-1})],$$

$$h(S'_k \backslash S'_{k-1}) = \sum_{l=r_{k-1}+1}^{r_k} h(S_l \backslash S_{l-1}).$$

(4) $t^{\#}(k) \triangleq \sqrt{(K(S'_k) - K(S'_{k-1}))/h(S'_k \backslash S'_{k-1})} \in [1/\sqrt{\beta}, \sqrt{\beta})t^*(r_k)$, for $k = 1, 2, \ldots, j$. This follows from property (3) and $t(l) \in [1/\sqrt{\beta}, \sqrt{\beta})t^*(r_k)$, for $l = r_{k-1} + 1, \ldots, r_k$, we have this property.

(5) $C_\beta(n, K, h, \beta_0) \geq 2\sum_{k=1}^{j} \sqrt{[K(S'_k) - K(S'_{k-1})]h(S'_k \backslash S'_{k-1})}$. Indeed, by properties (2) and (4), we have:

$$\begin{aligned} C_\beta(n, K, h, \beta_0) &= \sum_{k=1}^{j} \left\{ \frac{K(S'_k) - K(S'_{k-1})}{t^*(r_k)} + h(S'_k \backslash S'_{k-1})t^*(r_k) \right\} \\ &= \sum_{k=1}^{j} \sqrt{[K(S'_k) - K(S'_{k-1})]h(S'_k \backslash S'_{k-1})} \\ &\quad \times \left\{ \frac{t^{\#}(k)}{t^*(r_k)} + \frac{t^*(r_k)}{t^{\#}(k)} \right\} \\ &\geq 2 \sum_{k=1}^{j} \sqrt{[K(S'_k) - K(S'_{k-1})]h(S'_k \backslash S'_{k-1})}. \end{aligned}$$

Now we define a j-product instance (K', h') of the joint replenishment problem by letting $K'_k \triangleq K(S'_k)$, $K'(S) \triangleq \max_{k \in S} K'_k$, for all $S \subseteq N$, and $h'_k = h(S'_k \backslash S'_{k-1})$, for $k = 1, 2, \ldots, j$. Instance (K', h') has the following properties.

(1) $(K', h') \in (\mathscr{M}^j, \mathbb{R}^j_+)$.

(2) $(K', h') \in \mathscr{B}^j_\beta$. By property (4) of partition $\{S'_1, S'_2, \ldots, S'_j\}$, we have, for $k = 1, 2, \ldots, j$:

$$t'(k) \triangleq \sqrt{\frac{K'_k - K'_{k-1}}{h'_k}} = t^{\#}(k) \in [1/\sqrt{\beta}, \sqrt{\beta})\beta_0 \beta^{n_{r_k}},$$

where $K'_0 = 0$, all $n_{r_k} \in \mathbb{Z}$ and $n_{r_1} < n_{r_2} < \cdots < n_{r_j}$.

(3) $\mathrm{LB}(j, K', h') \leq \rho_\beta \,\mathrm{LB}(n, K, h)$. Indeed, by Corollary 2.2, property (5) for partition $\{S'_1, S'_2, \ldots, S'_j\}$ and Lemma 2.5:

$$\mathrm{LB}(j, K', h') = 2 \sum_{k=1}^{j} \sqrt{(K'_k - K'_{k-1})h'_k}$$

$$= 2 \sum_{k=1}^{j} \sqrt{[K(S'_k) - K(S'_{k-1})]h(S'_k \backslash S'_{k-1})}$$

$$\le C_\beta(n, K, h, \beta_0)$$

$$\le \rho_\beta \, \mathrm{LB}(n, K, h).$$

(4) $C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n) \le C_{\mathrm{GP}}(j, K', h', \mathscr{C}^n)$. Because to each grouping for j-product instance (K', h') corresponds a grouping for n-product instance (K, h) with the same average cost.

Based on these properties, for any $(K, h) \in (\mathscr{M}^n, \mathbb{R}^n_+)$, there exist $j \le n$ and $(K', h') \in \mathscr{B}^j_\beta$ such that

$$r(j, K', h', \mathscr{C}^n) = \frac{C_{\mathrm{GP}}(j, K', h', \mathscr{C}^n)}{\mathrm{LB}(j, K', h')}$$

$$\ge \frac{C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n)}{\rho_\beta \, \mathrm{LB}(n, K, h)}$$

$$= \frac{1}{\rho_\beta} r(n, K, h, \mathscr{C}^n).$$

Therefore, for any $(K, h) \in (\mathscr{M}^n, \mathbb{R}^n_+)$,

$$\max_{j \le n} r(j, \mathscr{B}^n_\beta, \mathscr{C}^n) \ge \frac{1}{\rho_\beta} r(n, K, h, \mathscr{C}^n),$$

$$\max_{j \le n} r(j, \mathscr{B}^n_\beta, \mathscr{C}^n) \ge \frac{1}{\rho_\beta} \sup_{(K,h) \in (\mathscr{M}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{C}^n),$$

$$\rho_\beta r^*_\beta \ge r^*.$$

This completes the proof. □

Lemma 4.3. *If* $1 \le y \le b$ *and* $0 < a_2 \le \sqrt{b}$, *then we have*

$$e\left(\frac{y}{a_2}\right) \le e\left(\frac{b}{a_2}\right).$$

Proof. Since $0 < a_2 \le \sqrt{b}$ and $1 \le b$, we have

$$1 \le \max\left\{a_2, \frac{1}{a_2}\right\} \le \frac{b}{a_2}.$$

As $e(x)$ is increasing for $x \ge 1$, based on the inequality above,

$$e\left(\frac{1}{a_2}\right) \le e\left(\max\left\{a_2, \frac{1}{a_2}\right\}\right) \le e\left(\frac{b}{a_2}\right).$$

Because $e(y/a_2)$ is a convex function of y and $1 \le y \le b$, we have

$$e\left(\frac{y}{a_2}\right) \le \max\left\{e\left(\frac{1}{a_2}\right), e\left(\frac{b}{a_2}\right)\right\} = e\left(\frac{b}{a_2}\right).$$

This completes the proof. □

Appendix B: Notation

The notation used in this paper is collected below and listed in alphabetical order.

- $\sum_{i=k+1}^{k} a_k \triangleq 0$, for any a, by convention.
- $\mathscr{B}_\beta^n \triangleq \{(K,h) \in \Omega^n \mid m_1(\beta_0) < m_2(\beta_0) < \cdots < m_n(\beta_0)$, for some $\beta_0 \in [1/\sqrt{\beta}, \sqrt{\beta})\}$ is the set of "root-of-β path" instances. Note that $m_i(\beta_0) \triangleq \lfloor \log_\beta(\sqrt{(K_i - K_{i-1})/h_i}\sqrt{\beta}/\beta_0) \rfloor$, $\forall i$.
- $C_{\mathrm{GP}}(n,K,h,\mathscr{C}^n) \triangleq \min_{G \in \mathscr{C}^n} C_{\mathrm{GP}}(n,K,h,G)$ is the optimal average cost of consecutive grouping heuristics for the n-product instance (K,h).
- $C_{\mathrm{GP}}(n,K,h,\mathscr{D}^n) \triangleq \min_{G \in \mathscr{D}^n} C_{\mathrm{GP}}(n,K,h,G)$ is the optimal average cost of grouping policies for the n-product instance (K,h).
- $C_{\mathrm{GP}}(n,K,h,G) \triangleq \inf_{T \in \mathbb{R}_+^{|G|}} C_{\mathrm{GP}}(n,K,h,G,T)$ is the optimal average cost of grouping policy G for the n-product instance (K,h).
- $C_{\mathrm{GP}}(n,K,h,G,T) \triangleq \sum_{i=1}^{|G|} \{K(G_i)/T_i + h(G_i)T_i\}$.
- $\mathscr{C}^n \triangleq$ the set of all consecutive partitions on N.
- $C_\beta(n,K,h,\beta_0) \triangleq \min_{t>0}\{\sum_{i=1}^{n} [(K(U_i) - K(U_{i-1}))/t_{\alpha_i} + h_{\alpha_i} t_{\alpha_i}]$, s.t. $t_i = \beta^{m_i}\beta_0$, $m_i \in \mathbb{Z}$, $\forall i \in N\}$ is the optimal average cost of power-of-β policies with fixed base period β_0 for the n-product instance (K,h). Note that α satisfies (2.1a) and the U_i's are defined by (2.1b).
- $C_\beta(n,K,h) \triangleq \min_{\beta_0>0} C_\beta(n,K,h,\beta_0)$ is the optimal average cost of power-of-β policies for the n-product instance (K,h).
- $\mathscr{D}^n \triangleq$ the set of all partitions on N.
- $e(x) \triangleq \frac{1}{2}(x + 1/x)$, for $x \in \mathbb{R}_+$, is the "EOQ sensitivity analysis function".
- $f_{jk}(t,T) \triangleq e(t_j/T_k) + \frac{1}{2}\sum_{i=k+1}^{|G|} t_j/T_i$, $\forall j \in G_k$, $k = 1,2,\ldots,|G|$.
- $G \triangleq \{G_1, G_2, \ldots, G_p\} \in \mathscr{D}^n$ is a *grouping* of the products in N.
- $|G| = p$ is the cardinality of set G, i.e., the number of groups in grouping G.
- $g(G_i) \triangleq 2\sqrt{K(G_i)h(G_i)}$ is the optimal average cost of group G_i.
- h_i is the holding cost rate for product $i \in N$.
- $h(S) \triangleq \sum_{i\in S} h_i$ is the holding cost rate for any set $S \subseteq N$.
- $K_i \triangleq K(\{i\})$ for all $i \in N$.
- $K(S)$ is the setup cost incurred when S is the set of all products being simultaneously replenished.
- $\mathrm{LB}(n,K,h) \triangleq \min_{t>0} \sum_{i=1}^{n} [(K(U_i) - K(U_{i-1}))/t_{\alpha_i} + h_{\alpha_i} t_{\alpha_i}]$ is the lower bound on all feasible policies. Note that α satisfies (2.1a) and the U_i's are defined by (2.1b).

- $\mathscr{M}^n \triangleq \{K: 2^N \mapsto \mathbb{R}_+ \mid 0 < K_1 \le K_2 \le \cdots \le K_n$ (nondecreasing), $K(S) = \max_{i \in S} K_i$, $\forall S \subseteq N$ (maximum)$\}$ denotes the set of "maximum" submodular set functions on N.
- n is the total number of products.
- $N \triangleq \{1, 2, \ldots, n\}$ is the set of products.
- $\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$ is the set of natural numbers.
- $p \triangleq |G|$ is the number of groups in grouping G.
- $r(n, K, h, \mathscr{C}^n) \triangleq C_{\mathrm{GP}}(n, K, h, \mathscr{C}^n)/\mathrm{LB}(n, K, h)$ is the optimal performance ratio of optimal consecutive grouping policies for the n-product instance (K, h).
- $r(n, K, h, \mathscr{D}^n) \triangleq C_{\mathrm{GP}}(n, K, h, \mathscr{D}^n)/\mathrm{LB}(n, K, h)$ is the optimal performance ratio of optimal grouping policies for the n-product instance (K, h).
- $r(n, K, h, G) \triangleq C_{\mathrm{GP}}(n, K, h, G)/\mathrm{LB}(n, K, h)$ is the performance ratio of grouping policy G for the n-product instance (K, h).
- $r(n, K, h, G, T) \triangleq C_{\mathrm{GP}}(n, K, h, G, T)/\mathrm{LB}(n, K, h)$ is the performance ratio of grouping policy G with replenishment period vector T for the n-product instance (K, h).
- $r^* \triangleq \sup_{n \in \mathbb{N}} r^*(n)$ is the worst-case performance ratio of grouping heuristics.
- $r^*(n) \triangleq \sup_{(K,h) \in (\mathscr{S}^n, \mathbb{R}^n_+)} r(n, K, h, \mathscr{D}^n)$ is the worst-case performance ratio of grouping heuristics over all n-product instances.
- $r^*_\beta \triangleq \sup_{n \in \mathbb{N}} \sup_{(K,h) \in \mathscr{B}^n_\beta} r(n, K, h, \mathscr{C}^n)$ is the worst-case performance ratio of consecutive grouping heuristics over all "root-of-β path" instances $(K, h) \in \mathscr{B}^n_\beta$.
- R_β is the worst-case performance ratio of power-of-β policies with fixed base period β_0.
- R^*_β is the worst-case performance ratio of power-of-β policies with variable base period β_0.
- $\mathbb{R}_+ \triangleq (0, \infty)$ is the set of positive real numbers.
- $\mathbb{R}^n_+ \triangleq \times_{i=1}^n \mathbb{R}_+$ is the set of n-dimensional positive real numbers.
- $\mathbb{R}^n_< \triangleq \{t \in \mathbb{R}^n_+ \mid t_1 < t_2 < \cdots < t_n\}$ is the "monotone cone" in $\mathbb{R}^n_+$.
- $\mathbb{R}^n_\beta \triangleq \{t \in \mathbb{R}^n_< \mid m_1(\beta_0) < m_2(\beta_0) < \cdots < m_n(\beta_0)$, for some $\beta_0 \in [1/\sqrt{\beta}, \sqrt{\beta})\}$ is the "root-of-β cone". Note that $m_i(\beta_0) \triangleq \lfloor \log_\beta(t_i(\sqrt{\beta}/\beta_0)) \rfloor, \forall i$.
- $\mathscr{S}^n \triangleq \{K: 2^N \mapsto \mathbb{R}_+ \mid K(\emptyset) = 0;\ K(S) \ge 0$, and $K(N) > 0$ (nonnegativity); $K(S) \le K(T)$, if $S \subseteq T$ (nondecreasing); $K(S \cap T) + K(S \cup T) \le K(S) + K(T)$, $\forall S, T \subseteq N$ (submodularity)$\}$ denotes the set of submodular set functions on N.
- $(S_1, S_2, \ldots, S_q)$ is a *nested path* of N satisfying $\emptyset \subset S_1 \subset S_2 \subset \cdots \subset S_q = N$.
- $t = (t_1, t_2, \ldots, t_n)$ is a replenishment period vector.
- $t^*_i \triangleq \beta_0 \beta^{\lfloor \log_\beta(t_i \sqrt{\beta}/\beta_0) \rfloor}, \forall i \in N$, is the replenishment period for product $i \in N$ in an optimal power-of-β policy to (JR) with β_0 fixed, where $t = (t_1, t_2, \ldots, t_n)$ is an optimal solution to (RJR).
- $T \triangleq (T_1, T_2, \ldots, T_p)$ is the replenishment period vector for grouping G.
- $T^*_i \triangleq \sqrt{K(G_i)/h(G_i)}$ is the optimal replenishment period for group G_i.
- $T^* \triangleq (T^*_1, T^*_2, \ldots, T^*_p)$ is the optimal replenishment period vector for grouping G.
- $W(n, t, G) \triangleq \inf_{T \in \mathbb{R}^{|G|}_+} \max_{j \in G_k,\, k=1,2,\ldots,|G|} f_{jk}(t, T)$.
- $W_\beta \triangleq \min_{a_1, a_2, b} \{\max_{i=1,\ldots,6} g_i(a_1, a_2, b) \mid 0 < a_1,\ 0 < a_2 \le \sqrt{b},\ 1 < b \le \sqrt{\beta}\}$.

- $\mathbb{Z} \triangleq \{0, \pm 1, \pm 2, \dots\}$ is the set of integers.
- $\rho_\beta \triangleq (1/\ln\beta)(\beta - 1)/\sqrt{\beta}$ is the upper bound on worst-case performance ratio of power-of-β policies.
- $\Omega^n \triangleq \{(K,h) \in (\mathscr{M}^n, \mathbb{R}^n_+) \mid \sqrt{K_1/h_1} < \sqrt{(K_2 - K_1)/h_2} < \cdots < \sqrt{(K_n - K_{n-1})/h_n}\}$ denotes the set of "monotone path" instances. Note the strict inequalities in the definition.

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